Elementary Theory of Russian Roulette

-interesting patters of fractions-

Satoshi Hashiba

Daisuke Minematsu

Ryohei Miyadera

Introduction. Today we are going to study mathematical theory of Russian roulette. If some people may feel bad about the Russian roulette game, we want to say sorry for them, but as a mathematical theory Russian roulette has a very interesting structure. We are sure that many of the reader can appreciate it. High school students made this theory with a little help by their teacher, so this article shows a wonderful possibility of a research by high school students.

In a Russian roulette game 2 persons play the game. They take turns and take up a gun and pull a trigger to themselves. The game ends when one of the players gets killed. Note that in this version of the game, one does not rotate the cylinder before he pulls the trigger.

In this article we often state the mathematica fact without proofs. Proofs are given at the appendix.

Problem 1. Suppose that we use a revolver with 6 chambers and 1 bullet. Calculate the probability of death of the first man mathematically.

Answer. Suppose that players A and B play the game, and A is the first player. In the first round A takes up the gun and pulls the trigger to himself. This time the probability of his death is $\frac{1}{6}$. If A survives, then in the second round B takes up the gun and does the same and if B survives, then in the third round A takes up the gun and does the same thing. Let's calculate the probability of A's death of third round. If A is to die in the third round, A has to survive the first round. The probability of survival for A in the first round is $\frac{5}{6}$, and after that B has to survive in the second round.

Since there are only 5 chambers and 1 bullet, so the probability of survival is $\frac{4}{5}$. Then there are 4 chambers and 1 bullet, and resulting probabily of death is $\frac{1}{4}$. Therefore the probability of death of A in the third round is $\frac{5}{6}\times\frac{4}{5}\times\frac{1}{4}$. As to the probability of A's death in the fifth round we can do the almost the same calculation and we get $\frac{5}{6}\times\frac{4}{5}\times\frac{3}{4}\times\frac{2}{3}\times\frac{1}{2}$. Finally the probability of death of the first player A is $\frac{1}{6}+\frac{5}{6}\times\frac{4}{5}\times\frac{1}{4}+\frac{5}{6}\times\frac{4}{5}\times\frac{3}{4}\times\frac{2}{3}\times\frac{1}{2}=\frac{3}{6}=\frac{1}{2}$.

Problem 2. Suppose that we use a revolver with 6 chambers and 2 bullets. Calculate the probability of death of the first man mathematically.

Answer. Since there are 2 bullets, the probability of his death in the first round is $\frac{2}{6}$, and probability of his survival is $\frac{4}{6}$. We can use almost the same method we used in problem 1 in the rest of the solution, and the answer is $\frac{2}{6}+\frac{4}{6}\times\frac{3}{5}\times\frac{2}{4}+\frac{4}{6}\times\frac{3}{5}\times\frac{2}{4}\times\frac{1}{3}\times\frac{2}{2}=\frac{9}{15}=\frac{3}{5}$.

So far we studied the case of 6 chambers, but in mathematics we can think of a gun with any number of chambers and bullets. For example we can study a gun with 100 chambers and 53 bullets. This is not a absurd idea even in a real life, because this may be a machine gun.

We denote by F[n,m] the probability of the first man's death when we use a revolver with n-chambers and m-bullets. For example by Problem 1 we have F[6,1] = $\frac{1}{2}$, and by Problem 2 F[6,2] = $\frac{3}{5}$.

Similarly we have F[n,m] $=\frac{m}{n}+\frac{n-m}{n}\times\frac{n-m-1}{n-1}\times\frac{m}{n-2}+\frac{n-m}{n}\times\frac{n-m-1}{n-1}\times\frac{n-m-2}{n-2}\times\frac{n-m-3}{n-3}\times\frac{m}{n-4}+\cdots$

By this formula you can calculate F[n,m] for any natural numbers n and m.

If you find this formula too difficult to understand, do not worry about it. If you can understand Problem 1 and 2, you can understand the rest of our article. You just have to understand that there is a way to calculate F[n,m] for any n and m.

Next is the best part of our article!

With F[n,m] for many n and m we make a triangle.

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                    {F[1,1]}        …Figure(1)
                {F[2,1],F[2,2]}
            {F[3,1],F[3,2],F[3,3]}
         {F[4,1],F[4,2],F[4,3],F[4,4]}
      {F[5,1],F[5,2],F[5,3],F[5,4],F[5,5]}
  {F[6,1],F[6,2],F[6,3],F[6,4],F[6,5],F[6,6]}
 {F[7,1],F[7,1],F[7,3],F[7,4],F[7,5],F[7,6],F[7,7]}
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By calculating F[n,m] we get the following triangle from the above triangle. Let's compare these 2 triangles. F[6,3] is the third in the 6th row of the above triangle. In the same position of the triangle below we have $\frac{13}{20}$. Therefore F[6,3]= $\frac{13}{20}$.

$$\{1\} \qquad \cdots\text{Figure (2)}$$
$$\left\{\frac{1}{2}, \frac{1}{1}\right\}$$
$$\left\{\frac{2}{3}, \frac{2}{3}, \frac{1}{1}\right\}$$
$$\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}\right\}$$
$$\left\{\frac{3}{5}, \frac{3}{5}, \frac{7}{10}, \frac{4}{5}, \frac{1}{1}\right\}$$
$$\left\{\frac{1}{2}, \frac{3}{5}, \frac{13}{20}, \frac{11}{15}, \frac{5}{6}, \frac{1}{1}\right\}$$
$$\left\{\frac{4}{7}, \frac{4}{7}, \frac{22}{35}, \frac{24}{35}, \frac{16}{21}, \frac{6}{7}, 1\right\}$$

Problem 3. Can you find any pattern in figure (2)?

Answer. Let's compare Figure (3) to the following Figure (2). If you reduce the fractions in Figure (3), the fractions generated will form Figure (2).

The pattern is quite obvious in Figure (3). For example look at 6th row. F[6,2] = $\frac{9}{15}$ and F[6,3] = $\frac{13}{20}$ are the second and third ones in the row. F[7,3] = $\frac{22}{35} = \frac{9+13}{15+20}$, which is the third one in the 7th row. This reminds us of Pascal's triangle. In general there exists the same kind of relation among F[n,m] ,F[n,m+1], F[n+1,m+1] for any natural number n and m with n ≥ m. For proof of the relation see Appendix 1.

$$\begin{array}{c}
\{1\} \quad \cdots\text{Figure(3)} \\
\{\frac{1}{2}, \frac{1}{1}\} \\
\{\frac{2}{3}, \frac{2}{3}, \frac{1}{1}\} \\
\{\frac{2}{4}, \frac{4}{6}, \frac{3}{4}, \frac{1}{1}\} \\
\{\frac{3}{5}, \frac{6}{10}, \frac{7}{10}, \frac{4}{5}, \frac{1}{1}\} \\
\{\frac{3}{6}, \frac{9}{15}, \frac{13}{20}, \frac{11}{15}, \frac{5}{6}, \frac{1}{1}\} \\
\{\frac{4}{7}, \frac{12}{21}, \frac{22}{35}, \frac{24}{35}, \frac{16}{21}, \frac{6}{7}, 1\}
\end{array}$$

Problem 4. Can you find any other pattern in figure (2)?

Answer. In fact there are several patterns. Please look at the following Figure (4). $\frac{F[4,3]+F[6,3]}{2} = (\frac{3}{4} + \frac{13}{20}) \div 2 = \frac{7}{10} = F[5, 3]$. In general we can prove that $\frac{F[2n,3]+F[2n+2,3]}{2}$ =F[2n+1,3]. As to the proof for this relation wee Appendix 2.

There are also other patterns. Look at the Figure (5) and (6). Can you find any pattern in Figure (5) and (6)?

$$\begin{array}{c}
\{1\} \qquad \cdots\text{Figure(4)} \\
\{\frac{1}{2}, \frac{1}{1}\} \\
\{\frac{2}{3}, \frac{2}{3}, \frac{1}{1}\} \\
\{\frac{2}{4}, \frac{4}{6}, \frac{3}{4}, \frac{1}{1}\} \\
\{\frac{3}{5}, \frac{6}{10}, \frac{7}{10}, \frac{4}{5}, \frac{1}{1}\} \\
\{\frac{3}{6}, \frac{9}{15}, \frac{13}{20}, \frac{11}{15}, \frac{5}{6}, \frac{1}{1}\} \\
\{\frac{4}{7}, \frac{12}{21}, \frac{22}{35}, \frac{24}{35}, \frac{16}{21}, \frac{6}{7}, 1\}
\end{array}$$

$$\begin{array}{c}
\{1\} \qquad \cdots\text{Figure(5)} \\
\{\frac{1}{2}, \frac{1}{1}\}
\end{array}$$

$$\begin{array}{c}
\left\{\frac{2}{3},\ \frac{2}{3},\ \frac{1}{1}\right\} \\
\left\{\frac{2}{4},\ \frac{4}{6},\ \frac{3}{4},\ \frac{1}{1}\right\} \\
\left\{\frac{3}{5},\ \frac{6}{10},\ \frac{7}{10},\ \frac{4}{5},\ \frac{1}{1}\right\} \\
\left\{\frac{3}{6},\ \frac{9}{15},\ \frac{13}{20},\ \frac{11}{15},\ \frac{5}{6},\ \frac{1}{1}\right\} \\
\left\{\frac{4}{7},\ \frac{12}{21},\ \frac{22}{35},\ \frac{24}{35},\ \frac{16}{21},\ \frac{6}{7},\ 1\right\}
\end{array}$$

$$\begin{array}{c}
\{1\} \qquad \cdots\text{Figure}(6) \\
\left\{\frac{1}{2},\ \frac{1}{1}\right\} \\
\left\{\frac{2}{3},\ \frac{2}{3},\ \frac{1}{1}\right\} \\
\left\{\frac{2}{4},\ \frac{4}{6},\ \frac{3}{4},\ \frac{1}{1}\right\} \\
\left\{\frac{3}{5},\ \frac{6}{10},\ \frac{7}{10},\ \frac{4}{5},\ \frac{1}{1}\right\} \\
\left\{\frac{3}{6},\ \frac{9}{15},\ \frac{13}{20},\ \frac{11}{15},\ \frac{5}{6},\ \frac{1}{1}\right\} \\
\left\{\frac{4}{7},\ \frac{12}{21},\ \frac{22}{35},\ \frac{24}{35},\ \frac{16}{21},\ \frac{6}{7},\ 1\right\}
\end{array}$$

Remark. We can also study the Russian Roulette game with more than 2 persons. For example if 4 persons play the game, then the probability of death of the third player form the following triangle.

Can you find any pattern in this triangle? Perhaps you will find this very similar to the Figure (3).

$$\begin{array}{c}
\left\{\frac{0}{1}\right\} \qquad \cdots\text{Figure}(7) \\
\left\{\frac{0}{2},\ \frac{0}{1}\right\} \\
\left\{\frac{1}{3},\ \frac{0}{3},\ \frac{0}{1}\right\} \\
\left\{\frac{1}{4},\ \frac{1}{6},\ \frac{0}{4},\ \frac{0}{1}\right\} \\
\left\{\frac{1}{5},\ \frac{2}{10},\ \frac{1}{10},\ \frac{0}{5},\ \frac{0}{1}\right\} \\
\left\{\frac{1}{6},\ \frac{3}{15},\ \frac{3}{20},\ \frac{1}{15},\ \frac{0}{6},\ \frac{0}{1}\right\} \\
\left\{\frac{2}{7},\ \frac{4}{21},\ \frac{6}{35},\ \frac{4}{35},\ \frac{1}{21},\ \frac{0}{7},\ \frac{0}{1}\right\}
\end{array}$$

Appendix 1. If you know combinatorics and how to calculate ${}_nC_m$, then you can read the proofs of mathematical facts presented in this article.

A proof for the fact presented at Problem 3.

To prove the existence of the relation of F[n,m] we need a different way to calculate F[n,m] from the way we used in Problem 1 and 2. Let me illustrate it by using problem 1.

Since we have 6 chambers, the chambers can be represented as {1,2,3,4,5,6} where we put 2 bullets, and there are ${}_6C_2$ ways to do that. The bullet which is in the chamber with a small number comes out first. If one bullet is in the chamber 1 and the other is in a chamber whose number is bigger than 1, then the first one will die. We have ${}_5C_1$ cases of this kind.

If one bullet is in the chamber 3 and the other is in a chamber whose number is bigger than 3, then the first man will die. We have ${}_3C_1$ cases of this kind.

If one bullet is in the chamber 5 and the other is in chamber 6, then the first man will die. We have ${}_1C_1$ case of this kind.

Therefore F[6,2] = $\frac{{}_5C_1+{}_3C_1+{}_1C_1}{{}_6C_2}$. Similarly we can prove that F[6,3]= $\frac{{}_5C_2+{}_3C_2}{{}_6C_3}$ and F[7,3] = $\frac{{}_6C_2+{}_4C_2+{}_2C_2}{{}_7C_3}$. By the famous equation ${}_pC_q = {}_{p-1}C_q + {}_{p-1}C_{q-1}$

${}_7C_3={}_6C_3+{}_6C_2$ and ${}_6C_2+{}_4C_2+{}_2C_2=({}_5C_2+{}_5C_1)+({}_3C_2+{}_3C_1)+{}_1C_1$, where we used a trivial fact that ${}_2C_2={}_1C_1$.

Therefore this is the reason of the existence of relation among F[6,2], F[6,3] and F[7,3].

Similarly we can prove that F[n,m]= $\frac{{}_{n-1}C_{m-1}+{}_{n-3}C_{m-1}+{}_{n-5}C_{m-1}+\cdots}{{}_nC_m}$, F[n,m+1]= $\frac{{}_{n-1}C_m+{}_{n-3}C_m+{}_{n-5}C_m+\cdots}{{}_nC_{m+1}}$ and F[n+1,m+1]= $\frac{{}_nC_m+{}_{n-2}C_m+{}_{n-4}C_m+\cdots}{{}_{n+1}C_{m+1}}$. By the equation ${}_pC_q = {}_{p-1}C_q + {}_{p-1}C_{q-1}$ we have ${}_{n+1}C_{m+1} = {}_nC_{m+1}+{}_nC_m$ and ${}_nC_m+{}_{n-2}C_m+{}_{n-4}C_m+\cdots=({}_{n-1}C_m+{}_{n-1}C_{m-1})+({}_{n-3}C_m+{}_{n-3}C_{m-1})+({}_{n-5}C_m+{}_{n-5}C_{m-1})+\cdots$.

Appendix 2. If You know how to calculate $\sum_{k=1}^n k^2$ and $\sum_{k=1}^n k$, then you can prove the fact presented at Problem 4, namely F[2n,3]+F[2n+2,3]=2F[2n+1,3].

A proof of a formula F[2n,3]+F[2n+2,3]=2F[2n+1,3]. F[2n,3] = $\frac{{}_{2n-1}C_2+{}_{2n-3}C_2+{}_{2n-5}C_2+\cdots}{{}_{2n}C_3}=\left(\sum_{k=1}^{n-1}\frac{(2n-2k+1)(2n-2k)}{2}\right)/\left(\frac{2n(2n-1)(2n-2)}{6}\right)$
$= 3\sum_{k=1}^{n-1}(2n^2-n(4k-1)+k(2k-1))/(n(2n-1)(2n-2))$

$= 3\sum_{k=1}^{n-1} ((2n^2 + n) - (4n + 1)k + 2k^2) / (n(2n - 1)(2n - 2))$
$= 3 \quad ((2n^2 + n)(n - 1) - (4n + 1) \times \frac{n(n-1)}{2} + 2 \times \frac{n(n-1)(2n-1)}{6} \quad) \quad / \quad ($
$n(2n - 1)(2n - 2))$
$= \frac{4n+1}{8n-4}$. Here if we put n+1 into n, we have
$F[2n+2,3] = \frac{4n+5}{8n+4}$.
$F[2n+1,3] = \frac{{}_{2n}C_2 + {}_{2n-2}C_2 + {}_{2n-4}C_2 + \cdots}{{}_{2n+1}C_3}$
$= 3\sum_{k=1}^{n} (2n - 2k + 2)(2n - 2k + 1) / (2n + 1) 2n (2n - 1)$
$= 3\sum_{k=1}^{n} ((4n^2 + 6n + 2) - (8n + 6)k + 4k^2) / ((2n + 1)(2n - 1) 2n)$
$= 3((4n^2 + 6n + 2)n - (8n + 6) \times \frac{n(n+1)}{2} + 4 \times \frac{n(n+1)(2n+1)}{6}) /$
$((2n + 1)(2n - 1) 2n)$
$= \frac{4n^2+3n-1}{8n^2-2}$.
Therefore we have F[2n,3]+F[2n+2,3]= 2F[2n+1,3].